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Journal of Algebra 322 (2009) 2848–2860

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Reflection groups acting on their hyperplanes

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ARTICLE INFO

Article history:
Received 21 January 2009
Available online 25 August 2009
Communicated by Michel Broué

Keywords:
Reflection arrangements
Reflection groups
Quadratic forms

ABSTRACT

After having established elementary results on the relationship between a finite complex (pseudo-)reflection group $W \subset \mathrm{GL}(V)$ and its reflection arrangement $\mathcal{A}$, we prove that the action of W on $\mathcal{A}$ is canonically related with other natural representations of W, through a 'periodic' family of representations of its braid group. We also prove that, when W is irreducible, then the squares of defining linear forms for $\mathcal{A}$ span the quadratic forms on V, which imply $|\mathcal{A}| \geqslant n(n+1)/2$ for $n = \dim V$, and relate the W-equivariance of the corresponding map with the period of our family.

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1. Introduction

Let V be finite-dimensional $\mathbb{C}$-vector space, $W \subset \mathrm{GL}(V)$ be a finite (pseudo-)reflection group with corresponding hyperplane arrangement $\mathcal{A}$. We assume that $\mathcal{A}$ is essential, meaning that $\bigcap \mathcal{A} = \{0\}$ and denote $n = \dim V$ the rank of W. We recall that an arrangement $\mathcal{A}$ is called irreducible if it cannot be written as $\mathcal{A}_1 \times \mathcal{A}_2$, and that W is called irreducible if it acts irreducibly on V. For details on the standard notions used in this introduction we refer to §2. A basic result can be written as follows

(0) $\mathcal{A}$ is irreducible iff W is irreducible.

Steinberg showed that the exterior powers of V are irreducible. His proof is based on the encoding of irreducibility by the connectedness of certain graphs. From this approach, the following is easily deduced

(1) If W is irreducible, then it contains an *irreducible* maximal parabolic subgroup.

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doi:10.1016/j.jalgebra.2009.08.005

Although this result is probably well known to experts and easily checked, it did not seem to have appeared in print earlier, and is a key tool for the sequel. The referee informed us that it can be found in the forthcoming book [LT].

We then consider the permutation W-module $\mathbb{C}\mathcal{A}$. A choice of linear maps $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$ defines a linear map $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ through $H \mapsto \alpha_H^2$. This map can be chosen to be a morphism of W-modules when W is a Coxeter group. We prove

(2) Φ is onto iff W is irreducible

meaning that each quadratic form on V is a linear combination of the quadratic forms α_H^2, as soon as W is irreducible. As a corollary, we get

(3) The cardinality of $\mathcal{A}$ is at least $n(n+1)/2$.

This lower bound is better than the usual $|\mathcal{A}| \geqslant n/2$ of [OT, Cor. 6.98], and is sharp, as $|\mathcal{A}| = n(n+1)/2$ when W is a Coxeter group of type A_n.

We denote d_H the order of the (cyclic) fixer in W of $H \in \mathcal{A}$, and define the distinguished reflection $s \in W$ to be the reflection in W with $H = \mathrm{Ker}(s-1)$ and additional eigenvalue $\zeta_H = \exp(2i\pi/d_H)$. We let $d : \mathcal{A} \to \mathbb{Z}$ denote $H \mapsto d_H$. We did not find the following in the standard textbooks:

(4) The data $(\mathcal{A}, d)$ determine W.

Letting B denote the braid group associated to W, we show that $\mathbb{C}\mathcal{A}$, considered as a linear representation of B, can be deformed through a path in $\mathrm{Hom}(B, \mathrm{GL}(V))$ which canonically connects $\mathbb{C}\mathcal{A}$ to other representations of W, including a faithful one. This turns out to provide a natural generalization of the action of Weyl groups on their positive roots to arbitrary reflection groups.

Finally, we prove that this path $h \mapsto R_h$ is periodic, namely that $R_{h+\kappa(W)} \simeq R_h$ for some integer $\kappa(W)$, with $\kappa(W) = 2$ when W is a Coxeter group. Moreover, $\kappa(W) = 2$ if and only if the morphism Φ above can be chosen to be a morphism of W-modules. In particular, we get

(5) If $\kappa(W) = 2$ then the W-module S^2V^* is a quotient of $\mathbb{C}\mathcal{A}$.

We emphasize the fact that the proofs presented here are elementary in the sense that, except for one of the last results, no use is made either of the Shephard–Todd classification of reflection groups, nor of the invariant theory of these groups.

2. Reflection groups and reflection arrangements

We recall from [OT] the following basic notions about reflection groups and hyperplane arrangements. An endomorphism $s \in \mathrm{GL}(V)$ is called a reflection if it has finite order and $\mathrm{Ker}(s-1)$ is a hyperplane of V. A finite subgroup W of some $\mathrm{GL}(V)$ which is generated by reflections is called a (complex) reflection group. The hyperplane arrangement associated to it is the collection $\mathcal{A}$ of the reflecting hyperplanes $\mathrm{Ker}(s-1)$ for s a reflection of W. There is a natural function $d : \mathcal{A} \to \mathbb{Z}$, $H \mapsto d_H$ which associates to each $H \in \mathcal{A}$ the order of the subgroup of W fixing H. We let $\zeta_H = \exp(2i\pi/d_H)$, and call a reflection s distinguished if its nontrivial eigenvalue is ζ_H, with $\mathrm{Ker}(s-1) = H$.

A nontrivial subgroup W_0 of W is called *parabolic* if it is the fixer of some linear subspace L of V. By a fundamental result of Steinberg, W_0 is a reflection subgroup of W. Moreover, if $\mathcal{A}_0 \subset \mathcal{A}$ is the smallest (possibly empty) collection of reflecting hyperplanes such that $L \subset \bigcap \mathcal{A}_0$, then the reflections of W_0 have for reflecting hyperplanes the elements of $\mathcal{A}_0$.

In general, a (central) hyperplane arrangement $\mathcal{A}$ is a finite collection of linear hyperplanes in V. When $\mathcal{A}$ originates from a reflection group W, then $\mathcal{A}$ is called a reflection arrangement. An arrangement $\mathcal{A}$ is called essential if $\bigcap \mathcal{A} = \{0\}$; for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ in V_1, V_2, the arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ in $V = V_1 \times V_2$ is defined as $\{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\}$; two arrangements in V

are isomorphic if there is an element of $\mathrm{GL}(V)$ that takes one to the other; an essential arrangement $\mathcal{A}$ is called irreducible if it is not isomorphic to some nontrivial $\mathcal{A}_1 \times \mathcal{A}_2$.

The following lemma shows that, when $\mathcal{A}$ is a reflection arrangement, the arrangement $\mathcal{A}$ together with the order of the reflections determines the reflection group. In particular, there is at most one reflection group with reflections of order 2 admitting a given reflection arrangement. Notice that $\mathcal{A}$ can be assumed to be essential, as the action of W on $\bigcap \mathcal{A}$ is necessarily trivial. Although basic, this fact does not appear in standard textbooks. The proof given here has been found in common with François Digne and Jean Michel.

Proposition 2.1. *Let $\mathcal{A}$ be an essential hyperplane arrangement in V.*

(1) *If $P \in \mathrm{GL}(V)$ satisfies $P(H) \subset H$ for all $H \in \mathcal{A}$, then P is semisimple.*
(2) *If $\mathcal{A}$ is a reflection arrangement associated to a complex reflection group $W \subset \mathrm{GL}(V)$, then $(\mathcal{A}, d)$ determines W.*

Proof. To prove (1), we choose linear forms $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$. Since $\mathcal{A}$ is essential, V^* is generated by the α_H, hence admits a basis made out some of them. The assumption then states that the α_H are eigenvectors for ${}^tP \in \mathrm{GL}(V^*)$, hence ${}^tP$ is semisimple and so is P. Now we prove (2), assuming that $W_1, W_2 \subset \mathrm{GL}(V)$ are two reflection groups with the same data $(\mathcal{A}, d)$. Let $H \in \mathcal{A}$ and $s_i \in W_i$ the distinguished reflection with $\mathrm{Ker}(s_i - 1) = H$. Then $x = s_1 s_2^{-1}$ fixes H and acts as 1 on V/H, hence is unipotent. The endomorphism $x \in \mathrm{GL}(V)$ clearly permutes the hyperplanes. Since $\mathcal{A}$ is finite, some power of x setwise stabilizes every $H \in \mathcal{A}$, hence is semisimple by (1). Since it is also unipotent this power of x is the identity, hence $x = 1$ because x is unipotent. It follows that $s_1 = s_2$ hence $W_1 = W_2$. □

3. A consequence of Steinberg's lemma

Let $W \subset \mathrm{GL}(V)$ be a reflection group and $\mathcal{A}$ the corresponding reflection arrangement. A basic fact is that the notions of irreducibility for W and $\mathcal{A}$ coincide and can be checked combinatorially on some graph. After recalling a proof of this, we notice a useful consequence.

We endow V with a W-invariant hermitian scalar product. Call $v \in V$ a *root* if it is an eigenvector of a reflection $s \in V$ such that $s.v \neq v$. For L a finite set of linearly independent roots we let V_L denote the subspace of V spanned by L, and Γ_L the graph on L connecting v_1 and v_2 if and only if v_1 and v_2 are not orthogonal. Notice that, if $s \in W$ is a reflection with root $v \in V$, the following properties hold for V_L, as they hold for any subspace: if $v \in V_L$ then $s(V_L) \subset V_L$, because $V_L = (\mathbb{C}v) \oplus (\mathrm{Ker}(s-1) \cap V_L)$; if $v \in V_L^{\perp}$ then $V_L \subset (\mathbb{C}v)^{\perp}$ is pointwise stabilized by s.

The following proposition is basic. We provide a proof of (1) ⇔ (2) for the convenience of the reader, because of a lack of reference. A proof of (1) ⇔ (3) in the case of well-generated reflection groups can be found in [Bo, Ch. V, Ex. 3] and is due to Steinberg. The extension given here actually shows that any irreducible reflection group contains an irreducible well-generated one.

Proposition 3.1. *The following are equivalent, for an essential reflection arrangement $\mathcal{A}$.*

(1) *W acts irreducibly on V.*
(2) *$\mathcal{A}$ is an irreducible hyperplane arrangement.*
(3) *V admits a basis L of roots such that Γ_L is connected.*

Proof. In the direction (2) ⇒ (1), if $V = V_1 \oplus V_2$ with the V_i being W-stable subspaces, then we define $\mathcal{A}_i = \{H \in \mathcal{A} \mid (s_H)_{|V_i} \neq 1\}$ with s_H the distinguished reflection w.r.t. $H \in \mathcal{A}$, and we have $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. In the direction (1) ⇒ (2), we let $V = V_1 \oplus V_2$ be the decomposition of V corresponding to $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. We choose a collection of roots for $\mathcal{A}$. Let s_1, s_2 be two distinguished reflections associated to $H_1 \in \mathcal{A}_1, H_2 \in \mathcal{A}_2$, respectively, and let $H = H_1 \oplus H_2 \subset V$. Consider some reflection $s \in W$ such that $\mathrm{Ker}(s-1) \supset H$. If $\mathrm{Ker}(s-1)$ can be written as $H_0 \oplus V_2$ with H_0 some hyperplane of V_1, then $H_0 \oplus V_2 \supset H_1 \oplus H_2$ implies $H_0 \supset H_1$, hence $H_0 = H_1$ by equality of dimensions, meaning that s is some power of s_1. Similarly, if $\mathrm{Ker}(s-1)$ can be written as $V_1 \oplus H_0$ with H_0 some hyperplane of V_2, then s is a power of s_2. Considering the reflection $s_2 s_1 s_2^{-1}$, which fixes H pointwise and has reflecting hyperplane $s_2.\mathrm{Ker}(s_1-1)$, since $s_1 \neq s_2$ it follows that $s_2 s_1 s_2^{-1}$ is a power of s_1. Then $s_2.\mathrm{Ker}(s_1-1) = \mathrm{Ker}(s_1-1)$ hence s_1, s_2 commute and have orthogonal roots. The subspace V_1^0 spanned by all roots arising from $\mathcal{A}_1$ is thus setwise stabilized by all reflections of W, hence $V_1^0 = V$. On the other hand, the hermitian scalar product induces an isomorphism between V_1^0 and V_1^* (because $\mathcal{A}_1$, like $\mathcal{A}$, is essential), hence $V_2 \neq \{0\} \Rightarrow V_1^0 \neq V$, a contradiction.

We now prove (1) $\Leftrightarrow$ (3). For A, B two sets or graphs we let $A \sqcup B$ denote their disjoint union. Let L_0 be of maximal size among the sets L of linearly independent roots with connected Γ_L. We prove that $|L| = \dim V$ if W is irreducible. Indeed, since W is irreducible generated by reflections and $V_{L_0} \subset V$, there would otherwise exist a reflection s such that $s(V_{L_0}) \not\subset V_{L_0}$. Letting $v \in V$ be a root of s, we have $v \notin V_{L_0}$ and $v \notin (V_{L_0})^\perp$. This proves that $L = L_0 \sqcup \{v\}$ is made out linearly independent roots and that Γ_L is connected, since $v \notin (V_{L_0})^\perp$ cannot be orthogonal to all roots spanning L_0 and L_0 is already connected. From this contradiction it follows that L_0 has cardinality $\dim V$. Conversely, if V admits a basis L of roots such that Γ_L is connected, then W is irreducible, for otherwise $V = V_1 \oplus V_2$ with V_1, V_2 nontrivial orthogonal W-stable subspaces, and $L = L_1 \sqcup L_2$ with $L_i = \{x \in L \mid x \in V_i\}$. Since L is a basis, the L_i are nonempty. Then $\Gamma_L = \Gamma_{L_1} \sqcup \Gamma_{L_2}$, contradicting the connectedness of Γ_L. □

Corollary 3.2. *If $W \subset \mathrm{GL}(V)$ is an irreducible reflection group then it admits an* irreducible *parabolic subgroup of rank* $\dim V - 1$.

Proof. Considering a set L of linearly independent roots such that Γ_L is connected, as given by the proposition, there exists $L_0 \subset L$ with $L = L_0 \sqcup \{v\}$ such that Γ_{L_0} is still connected. Then V_{L_0} has dimension $\dim V - 1$, and its orthogonal complement is spanned by some $v' \in V$. Letting W_0 denote the parabolic subgroup fixing v', it has rank $\dim V - 1$, admits for roots all elements of L_0, hence is irreducible since Γ_{L_0} is connected. □

4. Quadratic forms on *V*

Let $\mathcal{A}$ be an essential hyperplane arrangement in V. The integer $n = \dim V$ is the *rank* $\mathrm{rk}\,\mathcal{A}$ of $\mathcal{A}$. For each $H \in \mathcal{A}$ we let $\alpha_H \in V^*$ denote some linear form with kernel H. Let $\mathbb{C}\mathcal{A}$ denote the complex vector space with basis $v_H, H \in \mathcal{A}$, and define a linear map $\Phi : \mathbb{C}\mathcal{A} \to S^2 V^*$ by $\Phi(v_H) = \alpha_H^2$.

For Φ to be onto, it is necessary that $\mathcal{A}$ is irreducible. Indeed, if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ corresponds to some direct sum decomposition $V = V_1 \oplus V_2$, then choosing two nonzero linear forms $\varphi_i \in V_i^*$ defines a quadratic form $\varphi_1\varphi_2 \in S^2 V^*$ which does not belong to $\mathrm{Im}\,\Phi$. This condition is also sufficient in rank 2.

Proposition 4.1. *If $\mathcal{A}$ is essential of rank 2, then Φ is onto if and only if $\mathcal{A}$ is irreducible.*

Proof. Since $\mathcal{A}$ is essential, $\mathcal{A}$ contains at least two hyperplanes H_1, H_2. We denote $\alpha_i = \alpha_{H_i}$ the corresponding (linearly independent) linear forms. If $\mathcal{A} = \{H_1, H_2\}$, then $\mathcal{A}$ is obviously reducible, so we may assume that $\mathcal{A}$ contains at least another hyperplane. Let β denote the corresponding linear form. It can be written as $\beta = \lambda_1\alpha_1 + \lambda_2\alpha_2$ with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Since $\beta^2 = \lambda_1^2\alpha_1^2 + 2\lambda_1\lambda_2\alpha_1\alpha_2 + \lambda_2^2\alpha_2^2$ and $\alpha_1^2, \alpha_2^2, \beta^2 \in \mathrm{Im}\,\Phi$ we get $\alpha_1\alpha_2 \in \mathrm{Im}\,\Phi$. Since $\alpha_1^2, \alpha_2^2 \in \mathrm{Im}\,\Phi$ and α_1, α_2 are linearly independent it follows that $\mathrm{Im}\,\Phi = S^2 V^*$. □

This condition is not sufficient in rank 3, as shown by the following example. Consider in $\mathbb{C}^3$ the central arrangement of polynomial $xyz(x-y)(y-z)$. The morphism Φ is obviously not onto, as $\dim \mathbb{C}\mathcal{A} = 5$ and $\dim S^2 V^* = 6$. However, $\mathcal{A}$ is irreducible, because its Poincaré polynomial is $P_{\mathcal{A}}(t) = (1+t)(1+4t+4t^2)$, which is not divisible by $(1+t)^2$ – recall from [OT] that $P_{\mathcal{A}_1 \times \mathcal{A}_2} = P_{\mathcal{A}_1} P_{\mathcal{A}_2}$ and that $P_{\mathcal{A}}(t)$ is divisible by $1+t$ whenever $\mathcal{A}$ is central.

However, the condition is sufficient when $\mathcal{A}$ is a *reflection arrangement*.

Theorem 4.2. *Let $\mathcal{A}$ be a (essential) reflection arrangement. Then Φ is onto if and only if $\mathcal{A}$ is irreducible.*

Proof. We assume that $\mathcal{A}$ is irreducible, and prove that Φ is onto by induction on $\mathrm{rk}\,\mathcal{A}$. If $\mathrm{rk}\,\mathcal{A} \leqslant 2$, this is a consequence of the above proposition, so we can assume $\mathrm{rk}\,\mathcal{A} \geqslant 3$. We denote W the corresponding reflection group, and endow V with a W-invariant hermitian scalar product. By Corollary 3.2 there exists an irreducible maximal parabolic subgroup $W_0 \subset W$, defined by $W_0 = \{w \in W \mid w.v = v\}$ for some $v \in V \setminus \{0\}$. We let $H_0 = (\mathbb{C}v)^{\perp}$. By Steinberg's theorem W_0 is a reflection group, whose reflections are the reflections of W contained in W_0. Let $\mathcal{A}_0 \subset \mathcal{A}$ denote the arrangement in V corresponding to W_0. Since $v \in H$ for all $H \in \mathcal{A}_0$, by the induction hypothesis we have $S^2H_0^* \subset Q$, where $Q = \mathrm{Im}\,\Phi$ and $S^2H_0^* \subset S^2V^*$ is induced by $H_0^* \subset V^*$, this latter embedding being defined by letting $\gamma \in H_0^*$ act on $H_0^{\perp} = \mathbb{C}v$ by 0. Let $\alpha \in V^* \setminus \{0\}$ such that $H_0 = \mathrm{Ker}\,\alpha$. We have $S^2V^* = S^2H_0^* \oplus \alpha H_0^* \oplus \mathbb{C}\alpha^2$. Since $\mathcal{A}$ is irreducible, there exists $H \in \mathcal{A}$ such that $\alpha_H \notin \mathbb{C}\alpha$ and $\alpha_H \notin H_0^*$. Such a linear form can be written $\lambda(\alpha + \beta)$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\beta \in H_0^* \setminus \{0\}$. Then $(\alpha + \beta)^2 \in Q$ and $\beta^2 \in Q$, so we have $\alpha^2 + 2\alpha\beta \in Q$. We make W act on V^* by $w.\gamma(x) = \gamma(w^{-1}.x)$, for $x \in V$, $\gamma \in V^*$. Of course this action can be restricted to a W_0-action on $H_0^* \subset V^*$. Then $w.(\alpha + \beta)^2 \in Q$ for all $w \in W$, and since $w.\alpha = \alpha$ whenever $w \in W_0$, we get $\alpha^2 + 2\alpha(w.\beta) \in Q$ for all $w \in W_0$. Consider now the subspace U of H^* spanned by the $w_1.\beta - w_2.\beta$ for $w_1, w_2 \in W_0$. It is a W_0-stable subspace of H_0^*. Recall that H_0, hence H_0^*, is irreducible under the action of W_0. If $U = \{0\}$ then $w.\beta = \beta$ for all $w \in W_0$, hence $H_0^* = \mathbb{C}\beta$ by irreducibility of H_0^* and $\dim V = 2$, which has been excluded. Thus $U \neq \{0\}$ hence $U = H_0^*$. By $2\alpha(w_1.\beta - w_2.\beta) = (\alpha^2 + 2\alpha(w_1.\beta)) - (\alpha^2 + 2\alpha(w_2.\beta))$ we thus get $\alpha H_0^* \subset Q$. Then $(\alpha + \beta)^2 \in \alpha^2 + \alpha H_0^* + S^2H_0^* \subset \alpha^2 + Q$ implies $\alpha^2 \in Q$. It follows that $Q \supset S^2V^*$ which concludes the proof. $\square$

Corollary 4.3. *If $\mathcal{A}$ is an (essential) irreducible reflection arrangement of rank n, then $|\mathcal{A}| \geqslant n(n+1)/2$.*

Notice that the above lower bound is sharp, as it is achieved by Coxeter arrangements of type A_n.

When $\mathcal{A}$ is a reflection arrangement with corresponding reflection group W, both $\mathbb{C}\mathcal{A}$ and S^2V^* can be endowed with natural W-actions, where the action on $\mathbb{C}\mathcal{A}$ is defined by $w.v_H = v_{w(H)}$. It is thus natural to ask whether the linear forms α_H can be chosen such that Φ is a morphism of W-modules.

Proposition 4.4. *If $\mathcal{A}$ is a complexified* real *reflection arrangement (in particular W is a finite Coxeter group), then the linear forms α_H can be chosen such that Φ is a morphism of W-modules.*

Proof. We choose a W-invariant scalar product on the original real form V_0 of V and extend it to a W-invariant hermitian scalar product on V. For every $H \in \mathcal{A}$ we choose $x_H \in V_0$ orthogonal to H with norm 1, and define $\alpha_H : y \mapsto (x \mid y)$, our convention on hermitian scalar products being that they are linear on the right. Then, for any $w \in W$, $w.x_H \in V_0$ is orthogonal to $w(H)$ of norm 1, hence $w.x_H = \pm x_{w(H)}$. Since $w.\alpha_H$ maps y to $(w.x_H \mid y)$ we have $(w.\alpha_H)^2 = \alpha_{w(H)}^2$, which shows that Φ is a morphism of W-modules. $\square$

When W is not a Coxeter group, the W-modules $\mathbb{C}\mathcal{A}$ and S^2V^* are generally unrelated. However, this property is not a characterization of Coxeter groups, as there is at least one example of a (non-Coxeter) complex reflection group for which Φ can be a morphism of W-module. This is the group labelled G_{12} in the Shephard–Todd classification. Notice that, in such a case, one must have $\sum \alpha_H^2 = 0$, otherwise this sum would provide a copy of the trivial representation inside S^2V^*, forcing W to be a real reflection group.

We briefly describe this example. The group G_{12} can be described in $\mathrm{GL}_2(\mathbb{C})$ by 3 generators a, b, c of order 2, satisfying the relation $abca = bcab = cabc$. We choose the following model:

$$a = \begin{pmatrix} 1 & 1+\sqrt{-2} \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} -1 & 0 \\ 1-\sqrt{-2} & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} \sqrt{-2} & -1+\sqrt{-2} \\ -1-\sqrt{-2} & -\sqrt{-2} \end{pmatrix}.$$

We define a collection of vectors $e_H \in V$, such that $w.e_H = \pm e_{w(H)}$. Letting $\alpha_H : x \mapsto (e_H \mid x)$, the associated $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ is then a morphism of W-modules. A W-invariant hermitian scalar product is given on this matrix model by $(X \mid Y) = {}^t\bar{X}AY$ with

$$A = \begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}.$$

We choose for e_H the 12 following column vectors, which are fixed by the corresponding reflection s.

s ${}^te_H$	$babab$ $(1+\sqrt{-2}, -2)$	a $(1, 0)$	b $(0, 1)$
s ${}^te_H$	$ababa$ $(-2, 1-\sqrt{-2})$	bcb $(1, \sqrt{-2})$	c $(1, -1)$
s ${}^te_H$	$acaca$ $(1-\sqrt{-2}, 1+\sqrt{-2})$	cbc $(-1+\sqrt{-2}, -\sqrt{-2})$	aba $(-1-\sqrt{-2}, 1)$
s ${}^te_H$	bab $(-1, 1-\sqrt{-2})$	cac $(-\sqrt{-2}, 1+\sqrt{-2})$	aca $(-\sqrt{-2}, 1)$

It can be checked that the reflections a, b, c act on these vectors by monomial matrices, with nonzero entries in $\{\pm 1\}$ (hence factors through the hyperoctahedral group of rank 12). In this example, S^2V^* is a selfdual W-module.

For later reference, we state as a proposition the following remark.

Proposition 4.5. *For Φ to be a morphism of W-modules it is necessary that $\kappa(W) \leqslant 2$, where*

$$\kappa(W) = \min\bigl\{n \in \mathbb{Z}_{>0} \bigm| \forall w \in W\ \forall H \in \mathcal{A}\ w.\alpha_H = \zeta\alpha_H \Rightarrow \zeta^n = 1\bigr\}.$$

Using the Shephard–Todd classification, we will show in Section 6 that this condition is actually sufficient when W is irreducible.

5. A path between representations

In this section we define a natural connection between the action of W on $\mathbb{C}\mathcal{A}$ and more surprising representations of W. For this we need to introduce the space $X = V \setminus \bigcup \mathcal{A}$ of regular vectors, on which W acts freely, and its quotient (orbit) space X/W. We choose a base point $\underline{z} \in X$. The fundamental groups $B = \pi_1(X/W)$ and $P = \pi_1(X)$ are known as the braid group and pure braid group associated to W, respectively. There is a natural morphism $\pi : B \to W$ with kernel P. We first construct a deformation of $W \to \mathrm{GL}(\mathbb{C}\mathcal{A})$ as a linear representation of the braid group. This deformation should not be confused with the one introduced in [Ma07] when W is a 2-reflection group.

5.1. A representation of the braid group

To each $H \in \mathcal{A}$ is canonically associated a differential form $\omega_H = \frac{\mathrm{d}\alpha_H}{\alpha_H}$, using some arbitrary linear form α_H with kernel H. We introduce idempotents $p_H \in \mathrm{End}(\mathbb{C}\mathcal{A})$ defined by $p_{H_1}.v_{H_2} = v_{H_2}$ if $H_1 = H_2$, $p_{H_1}.v_{H_2} = 0$ otherwise. Choosing $h \in \mathbb{C}$, the 1-form

$$\omega = h \sum_{H \in \mathcal{A}} p_H \omega_H \in \Omega^1(X) \otimes \mathfrak{gl}(\mathbb{C}\mathcal{A})$$

satisfies $\omega \wedge \omega = 0$, hence defines a flat connection on the trivial vector bundle $X \times \mathbb{C}\mathcal{A} \to X$, which is clearly W-equivariant for the diagonal action on $X \times \mathbb{C}\mathcal{A}$. Dividing out by W, the corresponding flat bundle over X/W thus defines by monodromy a linear representation of B in $\mathbb{C}\mathcal{A}$. Letting

γ denote a representative loop of $\sigma \in B = \pi_1(X/W)$, we can lift it to a path $\tilde{\gamma}$ in X with endpoints $\underline{z}$ and $\pi(\sigma).\underline{z}$, where $\underline{z}$ is the chosen basepoint in X. The 1-forms $\tilde{\gamma}^*\omega_H$ can be written as $\gamma_H(t)\,\mathrm{d}t$ for some function γ_H on $[0,1]$, and the differential equation $\mathrm{d}f = (\gamma^*\omega)f$ to consider is then $f'(t) = h(\sum_{H\in\mathcal{A}} \gamma_H(t)p_H)f(t)$, with $f(0) = 1 \in \mathrm{End}(\mathbb{C}\mathcal{A})$. Since the p_H commute one to the other, the solution is easy to compute:

$$f(t) = \prod_{H\in\mathcal{A}} \exp\left(hp_H \int_0^t \gamma_H(u)\,\mathrm{d}u\right)$$

and the monodromy representation is given by

$$\sigma \mapsto R_h(\sigma) = \pi(\sigma) \prod_{H\in\mathcal{A}} \exp\left(hp_H \int_\gamma \omega_H\right)$$

where we identified $w \in W$ with $R_0(w) \in \mathrm{End}(\mathbb{C}\mathcal{A})$. In particular, the image of P is commutative. More precisely, if γ_0 is a loop in X around a single hyperplane H, the class $[\gamma_0] \in P$ is mapped to $\exp(2\mathrm{i}\pi hp_H)$. Since P is generated by such classes, it follows that $R_n(P) = \{1\}$ hence R_n factors through a representation of W whenever $n \in \mathbb{Z}$.

We recall that B is generated by so-called braided reflections (‘generators-of-the-monodromy’ in [BMR]), which are defined as follows. For a distinguished reflection $s \in W$, an element $\sigma \in B$ with $\pi(\sigma) = s$ is called a braided reflection if it admits as representative a path γ from $\underline{z}$ to $s.\underline{z}$ which is a composite $(s.\gamma_0)^{-1} * \gamma_1 * \gamma_0$ of paths with the following properties. Here $\gamma_0 : \underline{z} \rightsquigarrow \underline{z_0}$, $\gamma_1 : \underline{z_0} \rightsquigarrow s.\underline{z_0}$ and $(s.\gamma_0)^{-1} : s.\underline{z_0} \rightsquigarrow s.\underline{z}$ is the reverse path of $s.\gamma_0$, and $\gamma_1(t) = \exp(2\mathrm{i}\pi t/d_H)\underline{z_0}^- + \underline{z_0}^+$ where $\underline{z_0}^+$ and $\underline{z_0}^-$ are the orthogonal projection on H and $H^\perp$, respectively, and so that $\underline{z_0}^+ \notin H'$ for $H' \in \mathcal{A} \setminus \{H\}$.

Note that $\int_{s.\gamma_0} \omega_{H'} = \int_{\gamma_0} \omega_{s^{-1}(H')}$ for all $H' \in \mathcal{A}$, hence $\int_\gamma \omega_H = \int_{\gamma_1} \omega_H = (2\mathrm{i}\pi)/d_H$. In particular, for such a braided reflection σ we get

$$R_h(\sigma).v_H = \pi(\sigma)\exp\left(hp_H \int_\gamma \omega_H\right)v_H = \exp(2\,\mathrm{i}\pi h/d_H)v_H.$$

Moreover, if H and H' have orthogonal roots, then again $\int_\gamma \omega_{H'} = \int_{\gamma_1} \omega_{H'}$. But in this case $\alpha_{H'}(\gamma_1(t))$ is constant hence $\int_\gamma \omega_{H'} = 0$. An immediate consequence of this is that we can restrict ourselves to irreducible groups, namely

Proposition 5.1. *If $W = W_1 \times \cdots \times W_r$ is a decomposition of W into irreducible components, with corresponding decompositions $B = B_1 \times \cdots \times B_k$ and $\mathcal{A} = \mathcal{A}^1 \times \cdots \times \mathcal{A}^r$, then $R_h = R_h^{(1)} \times \cdots \times R_h^{(r)}$ with $R_h^{(k)} : B_k \to \mathrm{GL}(\mathbb{C}\mathcal{A}^k)$.*

It follows from the formulas above that, under the action of R_h, $\mathbb{C}\mathcal{A}$ is the direct sum of the stable subspaces $\mathbb{C}\mathcal{A}_k$, where $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_r$ is the decomposition of $\mathcal{A}$ into orbits under the action of W. We let $R_h^k : B \to \mathrm{GL}(\mathbb{C}\mathcal{A}_k)$, so that $R_h = R_h^1 \oplus \cdots \oplus R_h^r$.

Proposition 5.2. *If $h \notin \mathbb{Z}$, then R_h^k is irreducible for each $1 \leqslant k \leqslant r$.*

Proof. For each $H \in \mathcal{A}_k$ we choose a loop γ_H based at $\underline{z}$ around the hyperplane H. We have $\int_{\gamma_H} \omega_H = 2\mathrm{i}\pi$ and $\int_{\gamma_H} \omega_{H'} = 0$ for $H \neq H'$. Letting Q_H denote the class of γ_H in $P = \pi_1(X, \underline{z})$ we thus have $R_h^k(Q_H) = \exp(2\,\mathrm{i}\pi hp_H)$, hence $R_h^k(Q_H) - 1$ is a nonzero multiple of p_H if $h \notin \mathbb{Z}$. It follows that the elements $R_h^k(Q_H)$ generate the commutative algebra of diagonal matrices in $\mathrm{End}(\mathbb{C}\mathcal{A}_k)$. Let $\mathcal{G}_k$ be the oriented graph on the $v_H, H \in \mathcal{A}_k$ with an edge (v_{H_1}, v_{H_2}) if there exists $x \in B$ such that the matrix $R_h^k(x)$ has nonzero entry at (v_{H_1}, v_{H_2}). If $\mathcal{G}_k$ is connected, then R_h^k is irreducible (see e.g. [Ma04, Prop. 3, Cor. 2]). Choosing for each distinguished reflection $s \in W$ a braided reflection σ, $R_h^k(\sigma)$ has nonzero entries in $(v_H, v_{s(H)})$ and $(v_{s(H)}, v_H)$ for each $H \in \mathcal{A}$. Since $\mathcal{A}_k$ is an orbit under W and W is generated by distinguished reflections, it follows that $\mathcal{G}_k$ is connected, concluding the proof. □

Since R_h factors through W when $h \in \mathbb{Z}$, this has the following consequence.

Corollary 5.3. *For all $h \in \mathbb{C}$, the representation R_h of B is semisimple.*

We choose a collection of roots $e_H, H \in \mathcal{A}$. Notice that, for $w \in W$, $w(H) = H$ implies $w.e_H = e^{\mathrm{i}\theta} e_H$ for some $\theta \in \mathbb{R}$.

Lemma 5.4. *If $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ is a path in X with $w \in W$ such that $w.e_H = e^{\mathrm{i}\theta} e_H$, then $\int_\gamma \omega_H \in \mathrm{i}\theta + 2\mathrm{i}\pi\mathbb{Z}$.*

Proof. We can assume $-\pi < \theta \leqslant \pi$. Since $\int_\gamma \omega_H$ is independent of the choice of α_H, we can choose $\alpha_H : x \mapsto (e_H \mid x)$ with $(e_H \mid e_H) = 1$. We have $\alpha_H(w.x) = e^{\mathrm{i}\theta}\alpha(x)$. We write $\gamma(t) = \gamma_H(t) + \gamma_0(t) e_H$ with $\gamma_0 : [0,1] \to \mathbb{C}$ and $\gamma_H : [0,1] \to H$. Then $\alpha_H(\gamma(t)) = \gamma_0(t)$ and $\int_\gamma \omega_H = \int_{\gamma_0} \frac{\mathrm{d}z}{z}$. Letting $x = \alpha_H(\underline{z}) \in \mathbb{C}^\times$, we have $\gamma_0 : x \rightsquigarrow e^{\mathrm{i}\theta} x$. If $\gamma_1 : x \rightsquigarrow e^{\mathrm{i}\theta} x$ is an arbitrary path in $\mathbb{C}^\times$, then $\gamma_0 * \gamma_1^{-1}$ is a loop in $\mathbb{C}^\times$, hence $\int_{\gamma_0} \frac{\mathrm{d}z}{z} - \int_{\gamma_1} \frac{\mathrm{d}z}{z}$ is a multiple of $2\mathrm{i}\pi$. If $e^{\mathrm{i}\theta} = 1$ this concludes the proof. If $e^{\mathrm{i}\theta} = -1$ we consider $\gamma_1(t) = x e^{\mathrm{i}\pi t}$, for which $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \mathrm{i}\pi$. If $e^{\mathrm{i}\theta} = \zeta \notin \{1, -1\}$ we consider $\gamma_1(t) = (1-t)x + t e^{\mathrm{i}\theta} x$ and $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \log(1 + (e^{\mathrm{i}\theta} - 1)t)|_0^1$ where log denotes the natural determination of the logarithm over $\mathbb{C} \setminus \mathbb{R}^-$. It follows that $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \log e^{\mathrm{i}\theta} = \mathrm{i}\theta$, and the conclusion follows. □

We recall from Section 4 the definition of $\kappa(W)$.

$$\kappa = \kappa(W) = \min\left\{n \in \mathbb{Z}_{>0} \;\middle|\; \forall w \in W\ \forall H \in \mathcal{A}\ w.e_H = \zeta e_H \Rightarrow \zeta^n = 1\right\}$$

Theorem 5.5. *For all $h \in \mathbb{C}$, $R_{h+\kappa}$ is isomorphic to R_h. Moreover, κ is the smallest positive real number such that $R_\kappa \simeq R_0$.*

Proof. Recall from Corollary 5.3 that, for all $h \in \mathbb{C}$, R_h is semisimple. Letting χ_h denote the character of R_h on B, it is thus sufficient to prove $\chi_h = \chi_{h+\kappa}$ for all $h \in \mathbb{C}$ in order to get $R_{h+\kappa} \simeq R_h$. Let $g \in B$ with $w = \pi(g)$, and $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ a representing path. By the explicit formulas above, we have

$$\chi_h(g) = \sum_{w(H)=H} \exp\left(h \int_\gamma \omega_H\right)$$

and $R_{h+\kappa} \simeq R_h$ follows by Lemma 5.4. We now show that κ is minimal with this property. Assuming otherwise, we let $0 < h < \kappa$ such that $\chi_h = \chi_0$. By definition of κ there exists $w \in W$, $H \in \mathcal{A}$ such that $w.e_H = e^{\mathrm{i}\theta} e_H$ with $e^{\mathrm{i}\theta h} \neq 1$. Letting $g \in B$ with $\pi(g) = w$ and $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ a representing path, we have $\int_\gamma \omega_H \in \mathrm{i}\theta + 2\mathrm{i}\pi\mathbb{Z}$, hence $\exp(h \int_\gamma \omega_H) \neq 1$. It follows that $|\chi_h(g)| < \chi_0(g)$ hence a contradiction. □

5.2. New representations of W

When $n \in \mathbb{Z}$, the representation R_n of B factorizes through W. In case W is irreducible, the action of the center is easy to describe.

Lemma 5.6. *If $w \in W$ acts by $\lambda \in \mathbb{C}^\times$ on V, then $R_n(w) = \lambda^n$ if $n \in \mathbb{Z}$. More generally, if there exists $v \in X$ such that $w.v = \lambda v$ for some $\lambda \in \mathbb{C}^\times$, then $R_n(w)$ is conjugate to $\lambda^n R_0(w)$*

Proof. We first assume that w acts on V by λ. We can write $\lambda = \exp(\mathrm{i}\theta)$ with $0 < \theta \leqslant 2\pi$. We consider the loop $\gamma(t) = e^{\mathrm{i}\theta t}\underline{z}$ in X/W, whose image in W is w. By direct calculation we have $\int_\gamma \omega_H = \mathrm{i}\theta$ for all $H \in \mathcal{A}$ and the conclusion follows from the general formula for R_h. Now assume $w.v = \lambda v$ for some $\lambda = \exp(\mathrm{i}\theta)$ with $0 < \theta \leqslant 2\mathrm{i}\pi$. Up to conjugation, we can assume $v = \underline{z}$, the loop $\gamma(t) = e^{\mathrm{i}\theta t}\underline{z}$ in X/W has image w in W and we conclude as before. □

More involved tools prove the following.

Proposition 5.7. *If W_0 is a parabolic subgroup of W with hyperplane arrangement $\mathcal{A}$ and $n \in \mathbb{Z}$, then the restriction of R_n to W_0 is isomorphic to the direct sum of the representation R_n of W_0 and the permutation representation of W_0 on $\mathbb{C}(\mathcal{A} \setminus \mathcal{A}_0)$.*

Proof. We let R_h^0 denote the representation R_h for W_0 acting on $\mathbb{C}\mathcal{A}_0$, and S_h the direct sum of R_h^0 and the permutation representation of W_0 on $\mathcal{A} \setminus \mathcal{A}_0$. We can embed the braid group B_0 of W_0 inside B such that, as representations over $\mathbb{C}[[h]]$, the restriction to B_0 of R_h is isomorphic to S_h (see [Ma07, Theorem 2.9]). In particular, for all $g \in B_0$, the traces of $R_h(g)$ and $S_h(g)$ are equal, as formal series in h. Since these traces are holomorphic functions in h, it follows that they are equal for all $h \in \mathbb{C}$. This means that the semisimple representations of B_0 associated to the restriction of R_h and to S_h are isomorphic. Since the restriction of R_n and S_n are semisimple for all $n \in \mathbb{Z}$ the conclusion follows. □

In the special case of a parabolic subgroup fixing a hyperplane, we get the following consequence.

Corollary 5.8. *For any $H \in \mathcal{A}$ and $n \in \mathbb{Z}$, if σ is a braided reflection around H, then $R_n(\sigma)$ is conjugate to $R_0(\sigma)\exp((2n\mathrm{i}\pi/d_H)p_H)$.*

The determination of the action of the center enables us to prove that, contrary to R_0, R_1 is faithful in general.

Proposition 5.9.

(1) *R_0 has kernel $Z(W)$.*
(2) *R_1 is faithful on W.*
(3) $\mathrm{Ker}\, R_n = \{w \in Z(W) \mid w^n = 1\}$
(4) *For $h \notin \mathbb{Q}$, R_h induces a faithful representation of $B/(P,P)$.*

Proof. Without loss of generality (because of Proposition 5.1) we may assume that W is irreducible. Obviously (3) $\Rightarrow$ (2). Although (1) is also a special case of (3), we prove it separately. If $|\mathcal{A}| = 1$ the statement is obvious, so we assume $|\mathcal{A}| \geqslant 2$. Clearly $Z(W) \subset \mathrm{Ker}\, R_0$, as $\mathrm{Ker}(wgw^{-1} - 1) = w.\mathrm{Ker}(g-1)$ for all $g, w \in W$. Let $w \in W$ such that $R_0(w) = 1$, that is $w(H) = H$ for all $H \in \mathcal{A}$. Let $s \in W$ be a distinguished reflection with reflecting hyperplane H. Then wsw^{-1} is a reflection with $\mathrm{Ker}(wsw^{-1} - 1) = H$ which has the same nontrivial eigenvalue as s, hence $wsw^{-1} = s$. It follows that w commutes with all distinguished reflections of W, hence $w \in Z(W)$ since W is generated by such elements.

We now prove (3). Let $w \in \mathrm{Ker}\, R_n$. Since $R_n(w) = R_0(w)D$ for some diagonal matrix D, the nonzero entries of $R_n(w)$ determine the permutation matrix $R_0(w)$, hence $w \in Z(W)$. Since W is irreducible, w acts on V by some scalar $\lambda \in \mathbb{C}^\times$, hence $R_n(w) = \lambda^n = 1$ by Lemma 5.6, hence $w^n = 1$. The converse inclusion is clear by Lemma 5.6.

We finally prove (4). Let $b \in B/(P,P)$ and recall that $\pi : B \twoheadrightarrow W$ denotes the natural projection. If $b \notin P/(P,P)$, The same argument shows that $R_h(b) \neq 1$ if $\pi(b) \notin Z(W)$. Otherwise, if $\pi(b) = \exp(2i\pi/d) \in Z(W)$, then $R_h(b) = \exp(2i\pi h/d) \prod_{H\in\mathcal{A}} \exp(2i\pi h m_H p_H)$ for some collection $m_H \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$. In that case, for $h \notin \mathbb{Q}$, $R_h(b) = 1$ implies $m_H = -1$ for all $H \in \mathcal{A}$ and $d = 1$. In particular, $\pi(b) = \{1\}$ and $b \in P/(P,P)$ is the image of some $[\gamma] \in \pi_1(X)$ such that $(1/d) + m_H = (1/2i\pi)\int_\gamma \omega_H = 0$. Since $H_1(X)$ is a free $\mathbb{Z}$-module spanned by the ω_H this implies that the image of $[\gamma]$ inside $H_1(X)$ is zero, meaning that $b \in P/(P,P)$ equals 1. This proves that R_h is faithful on $B/(P,P)$ for $h \notin \mathbb{Q}$. □

Corollary 5.10. *The exponent of $Z(W)$ divides $\kappa(W)$. If W is irreducible then $|Z(W)|$ divides $\kappa(W)$.*

Proof. By the proposition, the period of the sequence $\operatorname{Ker} R_n$ is the exponent of $Z(W)$. Since $\operatorname{Ker} R_n$ is $\kappa(W)$-periodic the conclusion follows. If W is irreducible then $Z(W)$ is cyclic hence its order equals its exponent. □

In the proof of Theorem 5.5, we computed the character χ_n of R_n. We recall the result here:

Proposition 5.11. *For any $w \in W$ and $n \in \mathbb{Z}$ we have*

$$\chi_n(w) = \sum_{w.e_H = \zeta e_H} \zeta^n.$$

If $\tilde{K} = \mathbb{Q}(\zeta_d)$ is a cyclotomic field containing all eigenvalues of $R_1(W)$, then letting $c_n \in \operatorname{Gal}(\tilde{K} \mid \mathbb{Q})$ for $\gcd(n,d) = 1$ be defined by $c_n(\zeta_d) = \zeta_d^n$ we get from this proposition that $\chi_n = c_n \circ \chi_1$ for all n prime to d.

To illustrate this section we consider the example of W of type G_4. Let $j = \exp(2i\pi/3)$. The group W is generated by

$$s = \begin{pmatrix} 1 & -1 \\ 0 & j \end{pmatrix}, \qquad t = \begin{pmatrix} j & 0 \\ j & 1 \end{pmatrix}.$$

It is a reflection group of order 24, with two generators s, t of order 3 satisfying $sts = tst$, and center of order 2. We make the list of its irreducible representations. There are three 1-dimensional $S_\alpha : s, t \mapsto \alpha$, three 2-dimensional A_α with $\operatorname{tr} A_\alpha(s) = -\alpha$ for $\alpha \in \{1, j, j^2\}$, and a 3-dimensional one that we denote U. The reflection representation is A_{j^2}, and $\kappa(W) = 6$. From the character table of W one gets

$$\begin{aligned} R_0 &= S_1 + U & R_1 &= A_1 + A_{j^2} & R_2 &= S_{j^2} + U, \\ R_3 &= A_j + A_{j^2} & R_4 &= S_j + U & R_5 &= A_1 + A_{j^2}. \end{aligned}$$

5.3. *The case of Coxeter groups*

If W is a Coxeter group, we get a simpler form of the representation R_h. Recall that, in this case, $\mathcal{A}$ is the complexification of some real arrangement $\mathcal{A}_0$ in V_0, where V_0 is a real form of V; moreover, choosing some connected component $\mathcal{C}$ of $V_0 \setminus \bigcup \mathcal{A}_0$, called a Weyl chamber, determines n hyperplanes $H_1, \ldots, H_n$ called the *walls* of $\mathcal{C}$, and the corresponding n reflections $s_1, \ldots, s_n$ are called the simple reflections associated to $\mathcal{C}$. If $\underline{z} \in \mathcal{C}$, there is also a special set of generators for B, namely the braided reflections σ_i around H_i such that γ_0 is a straight (real) segment orthogonal to H_i. These are called the Artin generators of B (associated to the choice of a Weyl chamber).

Proposition 5.12. *If W is a Coxeter group with simple reflections $s_1, \ldots, s_n$, then $\sigma_i \mapsto R_0(s_i)\exp(i\pi h p_{H_i})$ defines a representation of B which is equivalent to R_h. In particular, R_1 is equivalent to a representation of W on $\mathbb{C}\mathcal{A}$ for which $s_i.v_H = v_{s(H)}$ is $H \neq H_i$, $s_i.v_{H_i} = -v_{H_i}$, and R_{h+2} is equivalent to R_h for any $h \in \mathbb{C}$, while $R_1 \not\simeq R_0$.*

Proof. We introduce the Weyl chamber $\mathcal{C} \subset V_0$ with respect to the simple reflections $s_1, \ldots, s_n$, with walls $H_i = \mathrm{Ker}(s_i - 1)$, $1 \leqslant i \leqslant n$. Up to conjugacy the base point $\underline{z}$ can be chosen inside the Weyl chamber, and we define roots $e_H \in V_0$ of norm 1 such that $\mathbb{C}e_H = \mathrm{Ker}(s-1)^\perp$ and $(e_H \mid \underline{z}) > 0$ for $\underline{z} \in \mathcal{C}$. We choose for α_H the linear form $x \mapsto (e_H \mid x)$. Let us denote $\log^+$ the complex logarithm on $\mathbb{C} \setminus \mathrm{i}\mathbb{R}_-^\times$, and define

$$D_h = \prod_{H \in \mathcal{A}} \exp\bigl(\mathrm{i}\pi p_H \log^+(e_H \mid \underline{z})\bigr).$$

We consider a simple reflection s_i around a wall H_i. Then the path γ representing σ_i can be chosen so that $(e_H \mid \gamma(t))$ has positive real part for each $t \in [0,1]$ and $H \neq H_i$. It follows that $t \mapsto \log^+(e_H \mid \gamma(t))$ has differential $\gamma^*\omega_H$ and $R_h(\sigma_i)$ equals

$$R_0(s_i) \prod_{H \in \mathcal{A}} \exp\left(hp_H \int_\gamma \omega_H\right) = R_0(s_i) \prod_{H \in \mathcal{A}} \exp\bigl(hp_H\bigl(\log^+(e_H \mid s_i.\underline{z}) - \log^+(e_H \mid \underline{z})\bigr)\bigr).$$

Moreover, $(e_H \mid s_i.\underline{z}) = (s_i.e_H \mid \underline{z}) = (e_{s_i(H)} \mid \underline{z})$ if $H \neq H_i$. Indeed, $s_i.e_H$ is orthogonal to $s_i(H)$ by W-invariance of the scalar product; since it has norm 1 and is real we only need to show that $(s_i.e_H \mid \underline{z}) > 0$. But $(s_i.e_H \mid \underline{z}) = (e_H \mid s_i.\underline{z})$ and the segment $[\underline{z}, s_i.\underline{z}]$ does not cross H, as s_i is a simple root and $H \neq H_i$, hence $s_i.e_H = e_{s_i(H)}$. If $H = H_i$, we have $(e_{H_i} \mid s_i.\underline{z}) = -(e_{H_i} \mid \underline{z})$. It follows that

$$R_h(\sigma_i) = s_i \exp(\mathrm{i}\pi hp_{H_i}) \prod_{H \in \mathcal{A} \setminus \{H_i\}} \exp\bigl(hp_H\bigl(\log^+(e_{s_0(H)} \mid \underline{z}) - \log^+(e_H \mid \underline{z})\bigr)\bigr)$$

namely

$$R_h(\sigma_i) = D_h s_i \exp(\mathrm{i}\pi hp_{H_i}) D_h^{-1}$$

for all $i \in [1,n]$, which concludes the proof. We have $R_1 \not\simeq R_0$ because $\mathrm{tr}\, R_1(s_1) = \mathrm{tr}\, R_0(s_1) - 1$. $\square$

The representation of W described in this proposition for $h = 1$ is natural in the realm of root systems. Indeed, if a set $\mathcal{P}$ of roots for $\mathcal{A}_0$ is chosen, such that $\mathcal{P}$ satisfies the axioms $(SR)_I$ and $(SR)_{II}$ of a root system (see [Bo]), and $\mathcal{P}$ is subdivided into positive and negative roots $\mathcal{P}^+, \mathcal{P}^-$ according to the chosen Weyl chamber, where $\mathcal{P}^+ = \{e_H, H \in \mathcal{A}\}$, then the representation described here is isomorphic to one on $\mathbb{C}\mathcal{P}^+$ described by $w.f_H = f_{w(H)}$ if $w.e_H \in \mathcal{P}^+$ and $w.f_H = -f_{w(H)}$ if $w.e_H \in \mathcal{P}^-$, where f_H denotes the basis element of $\mathbb{C}\mathcal{P}^+$ corresponding to $e_H \in \mathcal{P}^+$.

Finally, we notice that, when W is a Coxeter group, then the representation R_h for arbitrary h factorizes through the extended Coxeter group $\tilde{W} = B/(P,P)$ introduced by J. Tits in [Ti] (and denoted V there). This group $\tilde{W}$ is an extension of W by $P/(P,P) = P^{ab} \simeq H_1(X,\mathbb{Z}) \simeq \mathbb{Z}\mathcal{A}$, which is not split in general. Tits however showed (§2.7 in [Ti]) that $\tilde{W}$ embeds in the semidirect product $\tilde{W} \ltimes \mathbb{Z}\mathcal{A}$. Our construction gives a new proof of this fact.

Indeed, denote $\tilde{R}_h$ the representation equivalent to R_h that we defined in the previous proposition. Denoting y_H the generators of the group $\mathbb{Z}\mathcal{A}$, we have a representation $S_h : W \ltimes \mathbb{Z}\mathcal{A} \to \mathrm{GL}(\mathbb{Z}\mathcal{A})$ that maps y_H to $\exp(2\mathrm{i}\pi hp_H)$ and $w \in W$ to $\tilde{R}_1(w)$. This representation is clearly faithful for $h \notin \mathbb{Q}$. We have $\tilde{R}_{1+h}(\sigma_i) = \tilde{R}_1(\sigma_i)\exp(2\mathrm{i}\pi hp_{H_i})$, which shows that $\tilde{R}_{1+h}(\tilde{W}) \subset S_h(W \ltimes \mathbb{Z}\mathcal{A})$. Since $\tilde{R}_{1+h}$ is faithful on $\tilde{W}$ for $h \notin \mathbb{Q}$ by Proposition 5.9(4), this proves that $\tilde{W}$ embeds in $W \ltimes \mathbb{Z}\mathcal{A}$.

We give in Table 1 the decomposition into irreducibles of R_0, R_1 for the classical Coxeter groups of type A_n, B_n, D_n. As usual we label the irreducible representations of $\mathfrak{S}_n$ by partitions of size n (with the convention that $[n]$ is the trivial representation); we label the irreducible representations of type B_n by pairs of partitions (λ, μ) of total size n and we denote the restriction of (λ, μ) to the usual index 2 subgroup of type D_n by $\{\lambda, \mu\}$. Recall that $\{\lambda, \mu\} = \{\mu, \lambda\}$ is irreducible if and only if $\lambda \neq \mu$.

Table 1

	R_0
A_n, $n \geqslant 3$	$[n-1,2]+[n,1]+[n+1]$
B_n, $n \geqslant 4$	$([n-2,2],\emptyset)+([n-2],[2])+2([n-1,1],\emptyset)+2([n],\emptyset)$
B_3	$([1],[2])+2([2,1],\emptyset)+2([3],\emptyset)$
D_n, $n \geqslant 4$	$\{[n-2,2],\emptyset\}+\{[n-2],[2]\}+\{[n-1,1],\emptyset\}+\{[n],\emptyset\}$
	R_1
A_n, $n \geqslant 3$	$[n-1,1,1]+[n,1]$
B_n, $n \geqslant 3$	$([n-2,1],[1])+2([n-1],[1])$
D_n, $n \geqslant 4$	$\{[n-2,1],[1]\}+\{[n-1],[1]\}$

We sketch a justification of Table 1. For small values of n, we prove this by using the character table. Then we use induction with respect to a natural parabolic subgroup W_0 in the same series, for which the branching rule is well known. Restrictions of R_0 and R_1 to this parabolic subgroup are then isomorphic to the sum of the corresponding representation R_0 or R_1 of the subgroup, plus the permutation action of the reflections in W which do not belong to W_0 (this is clear for R_0, and a consequence of Proposition 5.7 for R_1). The decomposition into irreducibles of this permutation representation is easy, namely $[n-1,1]+[n]$ for A_n, $([n-2],[1])+([n-2,1],\emptyset)+2([n-1],\emptyset)$ for B_n and $\{[n-2],[1]\}+\{[n-2,1],\emptyset\}+\{[n-1],\emptyset\}$ for D_n. This provides the restrictions of R_0 and R_1 to W_0. From the combinatorial branching rule it is easy to check that, for say $n \geqslant 5$, only the given decompositions admit these restrictions.

6. Tables for $\kappa(W)$

We compute here the value of $\kappa(W)$ for all irreducible reflection groups W. More precisely, we compute all $d \in \mathbb{Z}$ such that there exists $w \in W$ and $H \in \mathcal{A}$ with $w.e_H = \zeta e_H$ and ζ of order d. We call these integers the $\mathcal{A}$-indices of W.

Recall that the group $G(de,e,r)$ for $r \geqslant 2$ is defined as the set of $r \times r$ monomial matrices with nonzero entries in $\mu_{de}(\mathbb{C})$, such that the product of these nonzero entries lie in $\mu_d(\mathbb{C})$.

Proposition 6.1. *The $\mathcal{A}$-indices of $W = G(de,e,r)$ are exactly the divisors of $\kappa(W)$. Moreover, $\kappa(W) = de$ if $d \neq 1$ or $r \geqslant 3$. If $W = G(e,e,2)$ then $\kappa(W) = 2$.*

Proof. Since $G(e,e,2)$ is a Coxeter (dihedral) group, we can assume $d \neq 1$ or $r \geqslant 3$. First note that the standard hermitian scalar product on $\mathbb{C}^r$ is invariant under W. We introduce the hyperplane arrangement

$$\mathcal{A}^0_{de,r} = \left\{ z_i - \zeta z_j = 0 \,\middle|\, \zeta \in \mu_{de}(\mathbb{C}) \right\}.$$

We have $\mathcal{A}^0_{de,r} \subset \mathcal{A}$, and the orthogonal complement to $H : z_i - \zeta z_j = 0$ is spanned by $e_H = e_i - \zeta^{-1} e_j$, if $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{C}^r$. Let $w \in W$. Since w is a monomial matrix, there exists $\lambda_1, \dots, \lambda_r \in \mu_{de}(\mathbb{C})$ with $\lambda_i \in \mu_{de}(\mathbb{C})$, $\prod \lambda_i \in \mu_d(\mathbb{C})$, and $\sigma \in \mathfrak{S}_r$ such that $w.e_i = \lambda_i e_{\sigma(i)}$. Then $w.e_H = \mu e_H$ iff $\lambda_i e_{\sigma(i)} - \lambda_j \zeta^{-1} e_{\sigma(j)} = \mu \lambda_i e_i + \mu \lambda_j e_j$. The two possibilities are $\mu = 1, \zeta = 1$ or $\mu \lambda_j = \lambda_i, \mu \lambda_i = \lambda_j \zeta^{-1}$, that is $\mu^2 = \zeta^{-1}, \mu = \lambda_i \lambda_j^{-1}$. It follows that $\mu \in \mu_{de}(\mathbb{C})$. Conversely, assume we choose $\mu \in \mu_{de}(\mathbb{C})$, and let $\zeta = \mu^{-2}$. If $r \geqslant 3$ we can define $w \in W$ by $\sigma = (1\ 2)$, $\lambda_2 = 1$, $\lambda_1 = \mu$, $\lambda_3 = \mu^{-1}$, $\lambda_k = 1$ for $k \geqslant 4$, and $w.e_H = \mu e_H$ for $H : z_1 - \zeta z_2 = 0$. We have $\mathcal{A} = \mathcal{A}^0_{de,r}$ when $d = 1$, so this settles this case and we can assume $d \neq 1$. In that case, $\mathcal{A} = \mathcal{A}^0_{de,r} \cup \mathcal{A}^+_r$, where $\mathcal{A}^+_r$ is made out the hyperplanes $H_i : z_i = 0$, whose orthogonal complements are spanned by the e_i. If $w.e_i = \mu e_i$ for $w \in W$ we obviously have $\mu \in \mu_{de}(\mathbb{C})$, and conversely if $\mu \in \mu_{de}(\mathbb{C})$ we can define $w \in W$ by $w.e_1 = \mu e_1$, $w.e_2 = \mu^{-1} e_2$ and $w.e_i = e_i$ for $i \geqslant 3$. It follows that in this case too the set of $\mathcal{A}$-indices is the set of divisors of de. $\square$

Table 2

ST	κ	ST	κ	ST	κ	ST	κ	ST	κ	ST	κ
4	6	10	12	16	10	22	4	28	2	34	6
5	6	11	24	17	20	23	2	29	4	35	2
6	12	12	2	18	30	24	2	30	2	36	2
7	12	13	8	19	60	25	6	31	4	37	2
8	4	14	6	20	6	26	6	32	6		
9	8	15	24	21	12	27	6	33	6		

By noticing that $G(2,1,r)$, $G(2,2,r)$ and $G(e,e,2)$, are Coxeter groups, this gives the following.

Corollary 6.2. *For $W = G(de,e,r)$, we have $\kappa(W) = 2$ iff W is Coxeter group, if and only if $de = 2$ or $(d,r) = (1,2)$.*

By checking out the 34 exceptional reflection groups, we prove case by case the following.

Proposition 6.3. *Let W be an irreducible complex reflection group. The set of $\mathcal{A}$-indices is exactly the set of divisors of $\kappa(W)$.*

Table 2 gives the value of $\kappa(W)$, where W is a complex reflection group labelled by its Shephard–Todd number (ST).

We remark that the only non-Coxeter irreducible reflection groups with $\kappa(W) = 2$ are G_{12} and G_{24}. As in the case of G_{12}, it is straightforward to check that it is possible to choose the 21 linear forms α_H such that the linear map $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ is a morphism of W-modules. This phenomenon is reminiscent of the special properties of their "root systems" in the sense of [Co]. We refer to [Sh, §2 and §4] for a detailed study of these special root systems of type G_{12} and G_{24}. In particular, convenient linear forms for G_{24} are described in [Sh, §4.1].

As a consequence of this case-by-case investigation, Propositions 4.4 and 4.5 can be enhanced to the following

Theorem 6.4. *Let W be an irreducible reflection group. The linear forms α_H can be chosen such that Φ is a morphism of W-modules if and only if $\kappa(W) = 2$. This is the case exactly when W is a Coxeter group or an exceptional reflection group of type G_{12} or G_{24}.*

References

[Bo] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4, 5, 6, Hermann, Paris, 1968.
[BMR] M. Broué, G. Malle, R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, J. Reine Angew. Math. 500 (1998) 127–190.
[Co] A.M. Cohen, Finite complex reflection groups, Ann. Sci. École. Norm. Sup. 9 (1976) 379–436.
[LT] G.I. Lehrer, D.E. Taylor, Unitary Reflection Groups, Cambridge University Press, in press.
[Ma04] I. Marin, Irréductibilité générique des produits tensoriels de monodromies, Bull. Soc. Math. France 132 (2004) 201–232.
[Ma07] I. Marin, Krammer representations for complex braid groups, preprint, arXiv:0711.3096, v3.
[OT] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer, Berlin, 1992.
[Sh] J.-Y. Shi, Simple root systems and presentations for certain complex reflection groups, Comm. Algebra 33 (2005) 1765–1783.
[Ti] J. Tits, Normalisateurs de tores I : Groupes de Coxeter étendus, J. Algebra 4 (1966) 96–116.